

Analytic Expressions for Geometric Measure of Three Qubit States

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A new method is developed to derive an algebraic equations for the geometric measure of entanglement of three qubit pure states. The equations are derived explicitly and solved in cases of most interest. These equations allow oneself to derive the analytic expressions of the geometric entanglement measure in the wide range of the three qubit systems, including the general class of W-states and states which are symmetric under permutation of two qubits. The nearest separable states are not necessarily unique and highly entangled states are surrounded by the one-parametric set of equally distant separable states. A possibility for the physical applications of the various three qubit states to quantum teleportation and superdense coding is suggested from the aspect of the entanglement.

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I. INTRODUCTION.

Entangled states have different remarkable applications and among them are quantum cryptography [1, 2], superdense coding [3, 4], teleportation [5, 6] and the potential speedup of quantum algorithms [7, 8, 9]. The entanglement of bipartite systems is well-understood [10, 11, 12, 13], while the entanglement of multipartite systems offers a real challenge to physicists. In contrast to bipartite setting, there is no unique treatment of the maximally entangled states for multipartite systems. In this reason it is highly difficult to formulate a theory of multipartite entanglement. Another point which makes difficult to understand the entanglement for the multi-qubit systems is mainly due to the fact that the analytic expressions for the various entanglement measures is extremely hard to derive.

We consider pure three qubit systems [14, 15, 16, 17], although the entanglement of mixed states attracts a considerable attention. For example, in recent experiment [18] the tangle for general mixed states was evaluated, which has never been done before. Three-qubit system is important in the sense that it is the simplest system which gives a non-trivial effect in the entanglement. Thus, we should understand the general properties of the entanglement in this system as much as possible to go further more complicated higher qubit system. The three-qubit system can be entangled in two inequivalent ways GHZ [19] and W, and neither form can be transformed into the other with any probability of success [20]. This picture is complete: any fully entangled state is SLOCC equivalent to either GHZ or W.

Only very few analytical results for tripartite entanglement

have been obtained so far [21] and we need more light on the subject. This is our main objective and we choose geometric measure of entanglement E_g [22, 23, 24, 25]. It is an axiomatic measure [22, 26, 27, 28], is connected with other measures [29, 30] and has an operational treatment. Namely, for the case of pure states it is closely related to the Groverian measure of entanglement [31] and the latter is associated with the success probability of Grover's search algorithm [32] when a given state is used as the initial state.

Geometric measure depends on entanglement eigenvalue Λ_{\max}^2 and is given by formula $E_g(\psi) = 1 - \Lambda_{\max}^2$. For pure states the entanglement eigenvalue is equal to the maximal overlap of a given state with any complete product state. The maximization over product states gives *nonlinear eigenproblem* [25] which, except rare cases, does not allow the complete analytical solutions.

Recently the idea was suggested that nonlinear eigenproblem can be reduced to the linear eigenproblem for the case of three qubit pure states [33]. The idea is based on theorem stating that any reduced $(n-1)$ -qubit state uniquely determines the geometric measure of the original n -qubit pure state. This means that two qubit mixed states can be used to calculate the geometric measure of three qubit pure states and this will be fully addressed in this work.

The method gives two algebraic equations of degree six defining the geometric measure of entanglement. Thus the difficult problem of geometric measure calculation is reduced to the algebraic equation root finding. Equations contain valuable information, are good bases for the numerical calculations and may test numerical calculations based on other numerical techniques [9].

Furthermore, the method allows to find the nearest separable states for three qubit states of most interest and get analytic expressions for their geometric measures. It turn out that highly entangled states have their own feature. Each highly entangled state has a vicinity with no product state and all nearest product states are on the boundary of the vicinity and

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form an one-parametric set.

In Section II we derive algebraic equations defining the geometric entanglement measure of pure three qubit states and present the general solution. In Section III we examine W-type states and deduce analytic expression for their geometric measures. States symmetric under permutation of two qubits are considered in Section IV, where the overlap of the state functions with the product states are maximized directly. In last Section V we make concluding remarks.

II. ALGEBRAIC EQUATIONS.

We consider three qubits A,B,C with state function $|\psi\rangle$. The entanglement eigenvalue is given by

$$\Lambda_{max} = \max_{q^1 q^2 q^3} |\langle q^1 q^2 q^3 | \psi \rangle| \quad (1)$$

and the maximization runs over all normalized complete product states $|q^1\rangle \otimes |q^2\rangle \otimes |q^3\rangle$. Superscripts label single qubit states and spin indices are omitted for simplicity. Since in the following we will use density matrices rather than state functions, our first aim is to rewrite Eq.(1) in terms of density matrices. Let us denote by $\rho^{ABC} = |\psi\rangle\langle\psi|$ the density matrix of the three-qubit state and by $\varrho^k = |q^k\rangle\langle q^k|$ the density matrices of the single qubit states. The equation for the square of the entanglement eigenvalue takes the form

$$\Lambda_{max}^2(\psi) = \max_{\varrho^1 \varrho^2 \varrho^3} \text{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \varrho^3). \quad (2)$$

An important equality

$$\max_{\varrho^3} \text{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \varrho^3) = \text{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2 \otimes \mathbb{1}^3) \quad (3)$$

was derived in [33] where $\mathbb{1}$ is a unit matrix. It has a clear meaning. The matrix $\text{tr}(\rho^{ABC} \varrho^1 \otimes \varrho^2)$ is $2 \otimes 2$ hermitian matrix and has two eigenvalues. One of eigenvalues is always zero and another is always positive and therefore the maximization of the matrix simply takes the nonzero eigenvalue. Note that its minimization gives zero as the minimization takes the zero eigenvalue.

We use Eq.(3) to reexpress the entanglement eigenvalue by reduced density matrix ρ^{AB} of qubits A and B in a form

$$\Lambda_{max}^2(\psi) = \max_{\varrho^1 \varrho^2} \text{tr}(\rho^{AB} \varrho^1 \otimes \varrho^2). \quad (4)$$

We denote by \mathbf{s}_1 and \mathbf{s}_2 the unit Bloch vectors of the density matrices ϱ^1 and ϱ^2 respectively and adopt the usual summation convention on repeated indices i and j . Then

$$\Lambda_{max}^2 = \frac{1}{4} \max_{s_1^2 = s_2^2 = 1} (1 + \mathbf{s}_1 \cdot \mathbf{r}_1 + \mathbf{s}_2 \cdot \mathbf{r}_2 + g_{ij} s_{1i} s_{2j}), \quad (5)$$

where

$$\mathbf{r}_1 = \text{tr}(\rho^A \boldsymbol{\sigma}), \quad \mathbf{r}_2 = \text{tr}(\rho^B \boldsymbol{\sigma}), \quad g_{ij} = \text{tr}(\rho^{AB} \sigma_i \otimes \sigma_j) \quad (6)$$

and σ_i 's are Pauli matrices. The matrix g_{ij} is not necessarily to be symmetric but must has only real entries. The maximization gives a pair of equations

$$\mathbf{r}_1 + g \mathbf{s}_2 = \lambda_1 \mathbf{s}_1, \quad \mathbf{r}_2 + g^T \mathbf{s}_1 = \lambda_2 \mathbf{s}_2, \quad (7)$$

where Lagrange multipliers λ_1 and λ_2 are enforcing unit nature of the Bloch vectors. The solution of Eq.(7) is

$$\mathbf{s}_1 = (\lambda_1 \lambda_2 \mathbb{1} - g g^T)^{-1} (\lambda_2 \mathbf{r}_1 + g \mathbf{r}_2), \quad (8a)$$

$$\mathbf{s}_2 = (\lambda_1 \lambda_2 \mathbb{1} - g^T g)^{-1} (\lambda_1 \mathbf{r}_2 + g^T \mathbf{r}_1). \quad (8b)$$

Now, the only unknowns are Lagrange multipliers, which should be determined by equations

$$|\mathbf{s}_1|^2 = 1, \quad |\mathbf{s}_2|^2 = 1. \quad (9)$$

In general, Eq.(9) give two algebraic equations of degree six. However, the solution (8) is valid if Eq.(7) supports a unique solution and this is by no means always the case. If the solution of Eq.(7) contains a free parameter, then $\det(\lambda_1 \lambda_2 \mathbb{1} - g g^T) = 0$ and, as a result, Eq.(8) cannot not applicable. The example presented in Section III will demonstrate this situation.

In order to test Eq.(8) let us consider an arbitrary superposition of W

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \quad (10)$$

and flipped W

$$|\widetilde{W}\rangle = \frac{1}{\sqrt{3}} (|011\rangle + |101\rangle + |110\rangle) \quad (11)$$

states, i.e. the state

$$|\psi\rangle = \cos \theta |W\rangle + \sin \theta |\widetilde{W}\rangle. \quad (12)$$

Straightforward calculation yields

$$\mathbf{r}_1 = \mathbf{r}_2 = \frac{1}{3} (2 \sin 2\theta \mathbf{i} + \cos 2\theta \mathbf{n}), \quad (13a)$$

$$g = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (13b)$$

where unit vectors \mathbf{i} and \mathbf{n} are aligned with the axes x and z , respectively. Both vectors \mathbf{i} and \mathbf{n} are eigenvectors of matrices g and g^T . Therefore \mathbf{s}_1 and \mathbf{s}_2 are linear combinations of \mathbf{i} and \mathbf{n} . Also from $\mathbf{r}_1 = \mathbf{r}_2$ and $g = g^T$ it follows that $\mathbf{s}_1 = \mathbf{s}_2$ and $\lambda_1 = \lambda_2$. Then Eq.(8) for general solution give

$$\mathbf{s}_1 = \mathbf{s}_2 = \sin 2\varphi \mathbf{i} + \cos 2\varphi \mathbf{n} \quad (14)$$

where

$$\sin 2\varphi = \frac{2 \sin 2\theta}{3\lambda - 2}, \quad \cos 2\varphi = \frac{\cos 2\theta}{3\lambda + 1}. \quad (15)$$

The elimination of the Lagrange multiplier λ from Eq.(15) gives

$$3 \sin 2\varphi \cos 2\varphi = \cos 2\theta \sin 2\varphi - 2 \sin 2\theta \cos 2\varphi. \quad (16)$$

Let us denote by $t = \tan \varphi$. After the separation of the irrelevant root $t = -\tan \theta$, Eq.(16) takes the form

$$\sin \theta t^3 + 2 \cos \theta t^2 - 2 \sin \theta t - \cos \theta = 0. \quad (17)$$

This equation exactly coincides with that derived in [25]. Since a detailed analysis was given in Ref.[25], we do not want to repeat the same calculation here. Instead we would like to consider the three-qubit states that allow the analytic expressions for the geometric entanglement measure by making use of Eq.(7).

III. W-TYPE STATES.

Consider W-type state

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle, \quad a^2 + b^2 + c^2 = 1. \quad (18)$$

Without loss of generality we consider only the case of positive parameters a, b, c . Direct calculation yields

$$\mathbf{r}_1 = r_1 \mathbf{n}, \quad \mathbf{r}_2 = r_2 \mathbf{n}, \quad g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -r_3 \end{pmatrix}, \quad (19)$$

where

$$r_1 = b^2 + c^2 - a^2, \quad r_2 = a^2 + c^2 - b^2, \quad r_3 = a^2 + b^2 - c^2 \quad (20)$$

and $\omega = 2ab$. The unit vector \mathbf{n} is aligned with the axis z . Any vector perpendicular to \mathbf{n} is an eigenvector of g with eigenvalue ω . Then from Eq.(7) it follows that the components of vectors \mathbf{s}_1 and \mathbf{s}_2 perpendicular to \mathbf{n} are collinear. We denote by \mathbf{m} the unit vector along that direction and parameterize vectors \mathbf{s}_1 and \mathbf{s}_2 as follows

$$\mathbf{s}_1 = \cos \alpha \mathbf{n} + \sin \alpha \mathbf{m}, \quad \mathbf{s}_2 = \cos \beta \mathbf{n} + \sin \beta \mathbf{m}. \quad (21)$$

Then Eq.(7) reduces to the following four equations

$$r_1 - r_3 \cos \beta = \lambda_1 \cos \alpha, \quad r_2 - r_3 \cos \alpha = \lambda_2 \cos \beta, \quad (22a)$$

$$\omega \sin \beta = \lambda_1 \sin \alpha, \quad \omega \sin \alpha = \lambda_2 \sin \beta, \quad (22b)$$

which are used to solve the four unknown constants $\lambda_1, \lambda_2, \alpha$ and β . Eq.(22b) impose either

$$\lambda_1 \lambda_2 - \omega^2 = 0 \quad (23)$$

or

$$\sin \alpha \sin \beta = 0. \quad (24)$$

First consider the case $r_1 > 0, r_2 > 0, r_3 > 0$ and coefficients a, b, c form an acute triangle. Eq.(24) does not give a true maximum and this can be understood as follows. If both vectors \mathbf{s}_1 and \mathbf{s}_2 are aligned with the axis z , then the last term in Eq.(5) is negative. If vectors \mathbf{s}_1 and \mathbf{s}_2 are antiparallel, then one of scalar products in Eq.(5) is negative. In this reason Λ_{\max}^2 cannot be maximal. Then Eq.(23) gives true maximum and we have to choose positive values for λ_1 and λ_2 to get maximum.

First we use Eq.(22a) to connect the angles α and β with the Lagrange multipliers λ_1 and λ_2

$$\cos \alpha = \frac{\lambda_2 r_1 - r_2 r_3}{\omega^2 - r_3^2}, \quad \cos \beta = \frac{\lambda_1 r_2 - r_1 r_3}{\omega^2 - r_3^2}. \quad (25)$$

Then Eq.(22b) and (23) give the following expressions for Lagrange multipliers λ_1 and λ_2

$$\lambda_1 = \omega \left(\frac{\omega^2 + r_1^2 - r_3^2}{\omega^2 + r_2^2 - r_3^2} \right)^{1/2}, \quad (26a)$$

$$\lambda_2 = \omega \left(\frac{\omega^2 + r_2^2 - r_3^2}{\omega^2 + r_1^2 - r_3^2} \right)^{1/2}. \quad (26b)$$

Eq.(7) allows to write a shorter expression for the entanglement eigenvalue

$$\Lambda_{\max}^2 = \frac{1}{4} (1 + \lambda_2 + r_1 \cos \alpha). \quad (27)$$

Now we insert the values of λ_2 and $\cos \alpha$ into Eq.(27) and obtain

$$4\Lambda_{\max}^2 = 1 + \frac{\omega \sqrt{(\omega^2 + r_1^2 - r_3^2)(\omega^2 + r_2^2 - r_3^2)} - r_1 r_2 r_3}{\omega^2 - r_3^2}. \quad (28)$$

The denominator in above expression is multiple of the area S of the triangle a, b, c

$$\omega^2 - r_3^2 = 16S^2. \quad (29)$$

A little algebra yields for the numerator

$$\begin{aligned} & \omega \sqrt{(\omega^2 + r_1^2 - r_3^2) + (\omega^2 + r_2^2 - r_3^2) - r_1 r_2 r_3} \\ &= 16 a^2 b^2 c^2 - \omega^2 + r_3^2. \end{aligned} \quad (30)$$

Combining together the numerator and denominator, we obtain the final expression for the entanglement eigenvalue

$$\Lambda_{\max}^2 = 4R^2, \quad (31)$$

where R is the circumradius of the triangle a, b, c . Entanglement value is minimal when triangle is regular, i.e. for W-state and $\Lambda_{\max}^2(W) = 4/9$ [25, 34].

Now consider the case $r_3 < 0$. Since $r_3 + r_1 = 2b^2 \geq 0$, we have $r_1 > 0$ and similarly $r_2 > 0$. Eq.(24) gives true maximum in this case and both vectors are aligned with the axis z

$$\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{n} \quad (32)$$

resulting in $\Lambda_{\max}^2 = c^2$. In view of symmetry

$$\Lambda_{\max}^2 = \max(a^2, b^2, c^2), \quad \max(a^2, b^2, c^2) > \frac{1}{2}. \quad (33)$$

Since the matrix g and vectors \mathbf{r}_1 and \mathbf{r}_2 are invariant under rotations around axis z the same properties must have Bloch vectors \mathbf{s}_1 and \mathbf{s}_2 . There are two possibilities:

i) Bloch vectors are unique and aligned with the axis z . The solution given by Eq.(32) corresponds to this situation and the resulting entanglement eigenvalue Eq.(33) satisfies the inequality

$$\frac{1}{2} < \Lambda_{\max}^2 \leq 1. \quad (34)$$

ii) Bloch vectors have nonzero components in xy plane and the solution is not unique. Eq.(21) corresponds to this situation and contains a free parameter. The free parameter is the angle defining the direction of the vector \mathbf{m} in the xy plane. Then Eq.(31) gives the entanglement eigenvalue in highly entangled region

$$\frac{4}{9} \leq \Lambda_{\max}^2 < \frac{1}{2}. \quad (35)$$

Eq.(31) and (33) have joint curves when parameters a, b, c form a right triangle and give $\Lambda_{\max}^2 = 1/2$. The GHZ states

have same entanglement value and it seems to imply something interesting. GHZ state can be used for teleportation and superdense coding, but W-state cannot be. However, the W-type state with right triangle coefficients can be used for teleportation and superdense coding [35]. In other words, both type of states can be applied provided they have the required entanglement eigenvalue $\Lambda_{\max}^2 = 1/2$.

IV. SYMMETRIC STATES.

Now let us consider the state which is symmetric under permutation of qubits A and B and contains three real independent parameters

$$|\psi\rangle = a|000\rangle + b|111\rangle + c|001\rangle + d|110\rangle, \quad (36)$$

where $a^2 + b^2 + c^2 + d^2 = 1$. According to Generalized Schmidt Decomposition [14] the states with different sets of parameters are local-unitary(LU) inequivalent. The relevant quantities are

$$\mathbf{r}_1 = \mathbf{r}_2 = r \mathbf{n}, \quad g = \begin{pmatrix} \omega & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

where

$$r = a^2 + c^2 - b^2 - d^2, \quad \omega = 2ad + 2bc \quad (38)$$

and the unit vector \mathbf{n} again is aligned with the axis z .

All three terms in the l.h.s. of Eq.(5) are bounded above:

- $\mathbf{s}_1 \cdot \mathbf{r}_1 \leq |r|$,
- $\mathbf{s}_2 \cdot \mathbf{r}_2 \leq |r|$,
- and owing to inequality $|\omega| \leq 1$, $g_{ij} s_{1i} s_{2j} \leq 1$.

Quite surprisingly all upper limits are reached simultaneously at

$$\mathbf{s}_1 = \mathbf{s}_2 = \text{Sign}(r)\mathbf{n}, \quad (39)$$

which results in

$$\Lambda_{\max}^2 = \frac{1}{2} (1 + |r|). \quad (40)$$

This expression has a clear meaning. To understand it we parameterize the state as

$$|\psi\rangle = k_1|00q_1\rangle + k_2|11q_2\rangle, \quad (41)$$

where q_1 and q_2 are arbitrary single normalized qubit states and positive parameters k_1 and k_2 satisfy $k_1^2 + k_2^2 = 1$. Then

$$\Lambda_{\max}^2 = \max(k_1^2, k_2^2), \quad (42)$$

i.e. the maximization takes a larger coefficient in Eq.(41). In bipartite case the maximization takes the largest coefficient in Schmidt decomposition [31, 36] and in this sense Eq.(41) effectively takes the place of Schmidt decomposition. When $|q_1\rangle = |0\rangle$ and $|q_2\rangle = |1\rangle$, Eq.(42) gives the known answer for generalized GHZ state [25, 34].

The entanglement eigenvalue is minimal $\Lambda_{\max}^2 = 1/2$ on condition that $k_1 = k_2$. These states can be described as follows

$$|\psi\rangle = |00q_1\rangle + |11q_2\rangle \quad (43)$$

where q_1 and q_2 are arbitrary single qubit normalized states. The entanglement eigenvalue is constant $\Lambda_{\max}^2 = 1/2$ and does not depend on single qubit state parameters. Hence one may expect that all these states can be applied for teleportation and superdense coding. It would be interesting to check whether this assumption is correct or not.

It turns out that GHZ state is not a unique state and is one of two-parametric LU inequivalent states that have $\Lambda_{\max}^2 = 1/2$. On the other hand W-state is unique up to LU transformations and the low bound $\Lambda_{\max}^2 = 4/9$ is reached if and only if $a = b = c$. However, one cannot make such conclusions in general. Five real parameters are necessary to parameterize the set of inequivalent three qubit pure states [14]. And there is no explicit argument that W-state is not just one of LU inequivalent states that have $\Lambda_{\max}^2 = 4/9$.

V. SUMMARY.

We have derived algebraic equations defining geometric measure of three qubit pure states. These equations have a de-

gree higher than four and explicit solutions for general cases cannot be derived analytically. However, the explicit expressions are not important. Remember that explicit expressions for the algebraic equations of degree three and four have a limited practical significance but the equations itself are more important. This is especially true for equations of higher degree; main results can be derived from the equations rather than from the expressions of their roots.

Eq.(7) give the nearest separable state directly and this separable states have useful applications. In order to construct an entanglement witness, for example, the crucial point lies in finding the nearest separable state [37]. This will be especially interesting for highly entangled states that have a whole set of nearest separable states and allow to construct a set of entanglement witnesses.

The expression in r.h.s. of Eq.(5) can be maximized directly for various three qubit states. Although it is very hard to solve the higher-degree equation, it turns out that the wide range of the three-qubit states have a symmetry and this symmetry reduces the equations of degree six to the quadratic equations. In this reason Eq.(5) can be used to derive the analytic expressions of the various entanglement measures for the three-qubit states. Also Eq.(5) can be a starting point to explore the numerical computation of the entanglement measures for the higher-qubit systems. We would like to discuss this issue elsewhere.

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